Norm and Operators on Partial Groupoids

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Abstract

The object of this paper is to introduce a new type of operation like a binary operation on a set, which is not completely defined on the set, but defined on two subsets of the set; the operation is a binary operation on these two subsets, with respect to which two subsets are separately groups and the set is completely definable with these two subsets uniquely - we call this operation as a partial binary operation on the set and the set equipped with this partial binary operation as partial groupoid. Scalar multiplication like linear space on a partial groupoid by the scalars of a field K is defined and a real valued function norm on a partial groupoid is also defined. A partial groupoid with a norm is termed as normed partial groupoid. It will be shown that a quasi-pseudo metric can be induced from a norm in a normed partial groupoid and the topology of a normed partial groupoid with respect to this quasi-pseudo metric will be studied. Lastly operators between two partial groupoid, their linearity etc are to be studied in this paper.

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1 Introduction and Preliminaries

It may sometime happens that we are not able to define a binary operation on a whole set, but we can define it on a subset or on some subsets of the set or we can only compose the elements of some subsets. Here we may think about the third case and suppose that operation is defined on two subsets and we can also combine two elements of one subset with the other subset. We will call it partial binary operation and the set equipped with a partial binary operation named as partial groupoid.

Usually norm or length concept is defined on a linear space, but we can define the norm or length or magnitude of a element in a group or partial groupoid. We will see in our study that this norm on a partial groupoid will generate a quasi-pseudo metric on the set and that will generate a topology on the normed partial groupoid - named as norm induce topology on the normed partial groupoid.

Like scalar multiplication with a vector in a linear space we can define the scalar multiplication type operation by the elements of a field on the partial groupoid can be defined. With a such type of scalar multiplication by the elements of a field K with the elements of a partial groupoid will be named as K-partial groupoid.

A linear operator between two vector or normed spaces over the same field K is a homomorphism between them with K linearity. Likewise we can also define homomorphism type operators between two partial groupoids and their linearity if both the partial groupoids are K-partial groupoid.

The symbol ■ will be use to indicate the end of any definition, example, theorem and its proof etc.

Before going to the main topic let us first recall some basic definitions:

Definition1.1.(Steen and Seebach[7]) A metric on a set X is a mapping:
d:$X \times X \rightarrow \mathbf{R}^+$ where $\mathbf{R}^+$ is the set of non-negative real numbers satisfying the following properties:
$\forall$ x,y,z$\in$X
M_1:d(x,x)=0
M_2:d(x,z)$\leq$d(x,y)+d(y,z)
M_3:d(x,y)=d(y,x)
M_4:if x$\neq$y, d(x,y)>0 .

We call d(x,y)the distance between x and y.

If d satisfies only M_1, M_2 and M_4 it is called quasi-metric.

If d satisfies only M_1, M_2 and M_3 it is called pseudo-metric.

If d satisfies only M_1 and M_2 it is called quasi-pseudo-metric. ■

Definition1.2.(Higgins[4]) A topological group is consisting of a set G, a binary operation o defined on G with respect to which G is a group and a topology defined on G such that:
(i) The group composition is continuous, i.e.
o:$G \times G \rightarrow G$, defined by (a,b)$\mapsto$aob is continuous. Here the topology on $G \times G$ is taken as the product topology.
(ii)The inversion operation is continuous, i.e.
ι:$G \rightarrow G$, defined by a$\mapsto a^{-1}$ is continuous.

2 Partial Groupoids and Norm on a Partial Groiupoid

Definition2.1. Let G be a non-empty set, G is said to be partial groupoid if there exist two non-empty subsets X and Y of G and a binary operation o not defined completely on G, but partially as o:$X \times X \rightarrow X$, o:$Y \times Y \rightarrow Y$ and o:$X \times Y \rightarrow G$ called partial binary operation on G induced by the order pair (X, Y) , such that:
(i) For each a$\in$G, there exist unique elements $a_1 \in$X and $a_2 \in$Y such that a=a_1oa_2, called the decomposition of under the partial binary operation o.
(ii)(X, o) and (Y, o) are groups.

We will denote the identity element of the groups (X, o) and (Y, o) by e_1 and e_2 respectively. The inverse of $a_1 \in$X and $a_2 \in$Y by ${a^{-1}}_1 \in$X and ${a^{-1}}_2 \in$Y respectively. We will always denote the unique decomposition of a$\in$G by a=$a_1 \in$o$a_2 \in$.
We will denote the above partial groupoid by (G, X, Y, o).■

Definition2.2. Let (G, X, Y, o) be partial groupoid, a system (H, P, Q) is said to be a sub partial groupoid of (G, X, Y, o) where $H \subseteq G$, P and Q are subsets of H as well as sub-group of X and Y respectively and when the partial binary operation o restricted on P and Q (H, P, Q, o) is also a partial groupoid.■

Theorem2.3.*In a partial groupoid (G, X, Y, o) if $X \cap Y \neq \emptyset$, then $e_1 = e_2 = e$ (say)and $X \cap Y = \{e\}$.*

Proof. Let $x \in X \cap Y$, i.e. $x \in X$ and $x \in Y$. So $x = e_1 o x = x o e_2$. By the uniqueness of decomposition $e_1 = x$ and $x = e_2$, i.e. $x = e_1 = e_2 = e$ (say). So $X \cap Y = \{e\}$, i.e. X and Y has the same identity element.■

§In the whole of our discussion, for any partial groupoid (G, X, Y, o) we will understand that $X \cap Y \neq \emptyset$, i.e. $e_1 = e_2 = e$ (say) and $X \cap Y = \{e\}$.

Definition2.4. Let (G, X, Y, o) be a partial groupoid and K be a field. Let us define a mapping:

$\bullet : K \times G \to G$ $(\lambda, a) \mapsto \bullet(\lambda, a) = \lambda a$ such that,
$\forall \lambda \in K$ and $a_1 \in X$, $a_2 \in Y$:
(i) $\lambda a_1 \in X$ and $\lambda a_2 \in Y$
(ii) $(\lambda a_1)^{-1} = \lambda a_1{}^{-1}$ and $(\lambda a_2)^{-1} = \lambda a_2{}^{-1}$.
(iii) $\lambda(a_1 o a_2) = (\lambda a_1) o (\lambda a_2)$.

We will call this partial groupoid with this mapping as (K, •) partial groupoid or simply K-partial groupoid. ■

Theorem2.5.*For a K-partial groupoid (G, X, Y, o), 0e=e, where 0 is the zero element in the field K.*

Proof. We have $e_1 = e_2 = e$ (say). $0eoe = 0e = 0(eoe) = 0eo0e$, using the left cancellation law in the group X or Y $0e = e$.■

Definition2.6. Let (G, X, Y, o) be a partial groupoid. A mapping
$\| \,.\, \| \; G \to \mathbf{R}$ $a \mapsto \|a\|$
is said to be norm on the partial groupid if it satisfies the following conditions:
$\forall$ a, b $\in G$:
(i) $\|a\| \geq 0$
(ii) $\|e_1\| = 0 = \|e_2\| = 0$, i.e. here $\|e\| = 0$
(iii) $\|(a_1 o b_1) o (a_2 o b_2)\| \leq \|a\| + \|b\|$, for $a = a_1 o a_2$ and $b = b_1 o b_2$, equally we can write condition as $\|(a_1 o b_1) o (a_2 o b_2)\| \leq \|(a_1 o a_2)\| + \|(b_1 o b_2)\|$.

The partial groupoid (G, X, Y, o) with the norm $\| \,.\, \|$ defined on G will be denoted by $[(G, X, Y, o), \| \,.\, \|]$ and this will be called a normed partial groupoid.

The norm is said to be symmetric about X (or Y) if for any $a_1 \in X$ (resp. $a_2 \in Y$) $\|a_1{}^{-1}\| = \|a_1\|$ (resp. $\|a_2{}^{-1}\| = \|a_2\|$). The norm is symmetric if it symmetric about X and Y respectively.

The norm is said to be totally symmetric if: for all $a_1 \in X$ and $a_2 \in Y$, $\|a_1^{-1} o a_2^{-1}\| = \|a_1 o a_2\|$.■

Theorem2.7.*For a normed partial groupoid [(G, X, Y, o), (i) ‖ . ‖], for a_1, $b_1 \in X$ and a_2, $b_2 \in Y$ $\|a_1 o a_2\| \leq \|a_1\| + \|a_1\|$. (ii) $\|a_1 o b_1\| \leq \|a_1\| + \|b_1\|$. (iii)$\|a_2 o b_2\| \leq \|a_2\| + \|b_2\|$.*

Proof. (i)We have $e_1=e_2=e$ (say). Putting $b_1=e_1$ and $b_2=e_2$ in the third condition of the definition of norm, the theorem follws.

(ii) $\|a_1 o b_1\| = \|(a_1 o b_1) o e_2\| = \|(a_1 o b_1) o (e_2 O e_2)\| \leq \|a_1 o e_2\| + \|b_1 o e_2\| = \|a_1\| + \|b_1\|$.

(iii) Similar as (ii).■

Example2.8.Let us consider $\mathbf{R}^2$, let X={(x,0):x∈ **R** and Y={(0,y):y∈ **R**. So X,Y⊂ **R**.

We define the partial binary operation "o" on $\mathbf{R}^2$ as $o:X\times Y \to \mathbf{R}^2$ defined as (x,0)o(0,y)=(x,y), $o:X\times X \to X$ defined as $(x_1,0)o(x_2,0) = (x_1+x_2,0)$ and $o:Y\times Y \to Y$ defined as $(0,y_1)o(0,y_2)=(0,y_1+y_2)$. ($\mathbf{R}^2$, X, Y, o) is a partial groupoid.

Now we define the mapping: ‖.‖ $\mathbf{R}^2 \to \mathbf{R}$, defined as $\|(x,y)\| = \sqrt{x^2+y^2}$, then ‖.‖ is a norm on the partial groupoid ($\mathbf{R}^2$, X, Y, o).■

Theorem2.9.*If ‖.‖ is totally symmetric norm on the partial groupoid (G, X, Y, o), then ‖.‖ is symmetric about X and Y both.*

Proof.We have $e_1=e_2=e$(say). Let $a_1 \in X$ and $a_2 \in Y$. $\|a_1\| = \|a_1 o e_1\| = \|a_1 o e_2\| = \|a_1^{-1} o e_2^{-1}\| = \|a_1^{-1} o e_2\| = \|a_1^{-1} o e_1\| = \|a_1^{-1}\|$.

Similarly,$\|a_2\| = \|a_2^{-1}\|$ and the theorem follows.■

§For a normed partial groupoid [(G, X, Y, o), ‖.‖], we define a binary operation $\odot: G\times G \to G$, defined as ∀ a, b∈G, $a \odot b = (a_1 o b_1) o (a_2 o b_2)$.

This mapping is well defined as the decompositions $a=a_1 o a_2$ and $b=b_1 o b_2$ are unique and we have the following theorem:

Theorem2.10.*(G, ⊙) is a group.*

Proof.The proof is easy and we leave the proof. It is to bee noted that the identity element in this group is $e_1 o e_2 = e$ and the inverse of $a=a_1 o a_2 \in G$ is $a^{/} = a_1^{-1} o a_2^{-1} \in G$.■

Corr2.11.*Both (X, o) and (Y, o) are commutative groups then (G, ⊙) is also a commutative group.*■

Corr2.12.(G, ⊙)is isomorphic to the product group of (X, o) and (Y, o).

Proof.We define the mapping $\pi: X\times Y \to G$, defined as $\pi(a_1,a_2)=a_1 o a_2$, which is a isomorphism from the product group of (X, o) and (Y, o) onto G and the theorem follows.■

§For a normed partial groupoid [(G, X, Y, o),‖ .‖] we define the mapping $d:G\times G \to \mathbf{R}$ defined as $d(a,b) = \|(a_1 o b_1^{-1}) o (a_2 o b_2^{-1})\|$.

With the above definition we have the following theorem:

Theorem2.13. *d is a quasi-psudo-metric on G, i.e. (G, d) is a quasi-psudo-metric space.*

Proof. $\forall$ a, b, c$\in$G, we see that:

(i) as $\|(a_1ob_1{}^{-1})o(a_2ob_2{}^{-1})\| \geq 0$, so $d(a,b)\geq 0$

(ii) $d(a,a)=\|(a_1oa_1{}^{-1})o(a_2oa_2{}^{-1})\|=\|(eoe)\|=\|e\|=0$.

(iii) $d(a,b)=\|(a_1ob_1{}^{-1})o(a_2ob_2{}^{-1})\|$

$=\|(a_1oc_1{}^{-1}oc_1ob_1{}^{-1})o(a_2ooc_2{}^{-1}oc_2ob_2{}^{-1})\|$

$=\|(a_1oc_1{}^{-1})o(a_2oc_2{}^{-1})\|+\|(c_1ob_1{}^{-1})o(c_2ob_2{}^{-1})\| =d(a,c)+d(c,b)$

Hence d is a quasi-psudo-metric on G, i.e. (G, d) is a quasi-psudo-metric space.■

Corr2.14. *If [(G, X, Y, o), ‖.‖] is totally symmetric the d is pseudo metric.* **Proof**. It is since $\forall$ a, b$\in$G,

$d(a,b)=\|(a_1ob_1{}^{-1})o(a_2ob_2{}^{-1})\|$

$=\|(a_1ob_1{}^{-1})^{-1}o(a_2ob_2{}^{-1})^{-1}\|$

$=\|(b_1oa_1{}^{-1})o(b_2oa_2{}^{-1})\| =d(b,a)$.■

Definition2.15. Let [(G, X, Y, o), ‖.‖] be a normed partial groupoid. We define a open ball with centre a$\in$G and radius r>0 by:

$B_r(a)=\{x\in G : d(x,a)<r\}$.

It is to be noted that each such ball is non-empty since at least $a\in B_r(a)$.

Now we have the following theorem:

Theorem2.16. *Let [(G, X, Y, o), ‖.‖] is symmetric about X (or Y). We restrict the metric d on X (or Y), then for a_1, b_1 $\in X$ (resp. a_2, b_2 $\in Y$) $d(a_1,b_1)=\|a_1ob_1{}^{-1}\|$ (resp. $d(a_2,b_2)=\|a_2ob_2{}^{-1}\|$). If $B_r{}^X(a_1)=\{x_1 \in X : \|x_1oa_1{}^{-1}\| <r\}$ (resp. $B_r{}^Y(a_2)=\{x_2 \in G : \|x_2oa_2{}^{-1}\| <r\}$) is a open ball with respected to the restricted norm on X (resp. in Y), then $B_r{}^X(a_1)=B_r(a_1)\cap X$ (resp. $B_r{}^Y(a_2)=B_r(a_2)\cap Y$).*

Proof. (We are proving for X only, the part for Y is similar).

We have $e_1=e_2=e$ (say).

Now, $a_1=a_1oe_2$ and $b_1=b_1oe_2$, so $d(a_1,b_1)=\|(a_1ob_1{}^{-1})o(e_2oe_2{}^{-1})\|$

$=\|(a_1ob_1{}^{-1})oe_2\|=\|(a_1ob_1{}^{-1})oe_1\|$

$=\|a_1ob_1{}^{-1}\|$.

Again we see that, if $x_1 \in B_r{}^X(a_1)$, then $x_1 \in X$ and $\|x_1oa_1{}^{-1}\| <r$, but $d(x_1,a_1)=\|x_1oa_1{}^{-1}\| <r$, i.e. $x_1 \in B_r(a_1)$ and hence, $B_r{}^X(a_1)\subseteq B_r(a_1)\cap X$.

Again let, $x\in B_r(a_1)\cap X$, so $x=x_1$(say)$\in X$ and $x=x_1 \in B_r(a_1)$. Now $a=a_1=a_1oe_2$ and $x=x_1=x_1oe_2$. Since $x_1 \in B_r(a_1)$, so $d(x,a) =\|(x_1oa_1{}^{-1})o(e_2oe_2{}^{-1})\|<r$, i.e. $\|x_1oa_1{}^{-1}\|<r$, so $x\in B_r{}^X(a_1)$.

Hence $x\in B_r(a_1)\cap X \Rightarrow x\in B_r{}^X(a_1)\cap X$, i.e. $B_r(a_1)\cap X \subseteq B_r{}^X(a_1)$.

So $B_r(a_1)\cap X = B_r{}^X(a_1)$.■

Theorem2.17. *The collection Σ={ $B_r(a)$: a$\in$G, r>0 } forms a basis for a topology on G.*

Proof. We see that : d(a,a)=0, so for any r>0 and a$\in$G, $a\in B_r(a)$.

Hence $G=\bigcup_{a\in G}B_r(a)$.

So, (i) There is a sub collection of Σ whose union is G.

(ii) Let a, b$\in$G and r, s>0 and $B_r(a)\cap B_s(b)\neq \phi$.

Let $c\in B_r(a)\cap B_s(b)$.

So $c\in B_r(a)$ and $c\in B_s(b)$,

i.e. d(c,a)<r and d(c,b)<s.

Let t=min{ r-d(c,a),s-d(c,b)}, then t>0.

Let us consider the ball $B_t(c)$,

then, $c \in B_t(c) \subseteq B_r(a) \cap B_s(b)$.

(i) and (ii) shows that Σ forms base for a topology $\tau(\Sigma)$(say) on G, we will say this topology as the norm induced topology on G.■

Corr2.18. *The colection* { $B_r{}^X(a_1)$: $a_1 \in X$, $r>0$ } *forms the subspace topology on* $X \subseteq G$ *and* { $B_r{}^Y(a_2)$: $a_2 \in Y$, $r>0$ } *forms the subspace topology on* $Y \subseteq G$.■

Theorem2.19.*For a normed partial groupoid [(G, X, Y, o), ‖.‖] the norm is symmetric about X (or Y) then the induced quasi-pseudo metric on X (resp. Y) from the quasi-pseudo metric on G induced from the norm is a pseudo-metric on X (resp. on Y).*

Proof. (We are doing for X only, the other part for Y is similar).

This is since for a_1, $b_1 \in X$,

$d(a_1,b_1)=\|a_1ob_1{}^{-1}\|=\|(a_1ob_1{}^{-1})^{-1}\| =\|b_1oa_1{}^{-1}\|=d(b_1,a_1)$.■

Theorem2.20. *In a normed partial groupoid [(G, X, Y, o), ‖.‖],if the norm is symmetric about X (or Y) and (X, o) (resp. (Y, o)) is commutative group then X (resp. Y) is a topological group.*

Proof. (We doing the proof for X only, the proof is similar for Y).

(i) Let a_1, $b_1 \in X$. For any r>0 we consider the ball $B_r(a_1ob_1)$ about a_1ob_1 in X. Again we consider the open balls $B_{r/2}{}^X(a_1)$ and $B_{r/2}{}^X(b_1)$ about a_1 and b_1 respectively in X, so $B_{r/2}{}^X(a_1)\times B_{r/2}{}^X(b_1)$ is a open set in X×X.

For $(x_1,y_1)\in B_{r/2}{}^X(b_1)\times B_{r/2}{}^X(b_1)$, $x_1 \in B_{r/2}{}^X(b_1)$ and $y_1 \in B_{r/2}{}^X(b_1)$.

i.e. $d(x_1,a_1)=\|x_1oa_1{}^{-1}\| <r/2$ and $d(y_1,b_1)=\|y_1ob_1{}^{-1}\| <r/2$.

Now,$x_1oy_1 \in X$ and

$d(x_1oy_1,a_1ob_1)=\|(x_1oy_1)o(a_1ob_1)^{-1}\| =\|(x_1oy_1)o(b_1{}^{-1}oa_1{}^{-1})\|, =\|(x_1oa_1{}^{-1})o(y_1ob_1{}^{-1})\|$,

(Since (X, o) is a commutative group).

$=\|x_1oa_1{}^{-1}\|+\|y_1ob_1{}^{-1}\| <r/2+r/2=r$, . So, $x_1oy_1 \in B_r{}^X(a_1ob_1)$,

i.e. $(x_1,y_1)\in B_{r/2}{}^X(b_1)\times B_{r/2}{}^X(b_1)\Rightarrow x_1oy_1 \in B_r{}^X(a_1ob_1)$,

i.e. the mapping o:X×X→X , defined as $(x_1,y_1)\mapsto x_1oy_1$ is continuous.

(ii) Let $a_1 \in X$. For any r>0 we consider the ball $B_r{}^X(a_1{}^{-1})$ about in $a_1{}^{-1} \in X$. Again we consider the open ball $B_r{}^X(a_1)$about a_1 in X.

For $x_1 \in B_r{}^X(a_1)$, $\|x_1oa_1{}^{-1}\| <r$.

Now, $\|x_1{}^{-1}o(a_1{}^{-1})^{-1}\|=\|x_1{}^{-1}oa_1\| =\|(x_1{}^{-1}oa_1)^{-1}\|$,

(Since ‖.‖ is symmetric about X)

$=\|a_1{}^{-1}ox_1\|=\|x_1oa_1{}^{-1}\| <r$

(Since (X, o) is commutative group)

Hence, $x_1 \in B_r{}^X(a_1)\Rightarrow x_1{}^{-1} \in B_r{}^X(a_1{}^{-1})$,

i.e. the mapping X→X defined by $x_1 \mapsto x_1{}^{-1}$ is continuous.

(i) and (ii) shows that X is a topological group.■

Theorem2.21. *In the normed partial groupoid [(G, X, Y, o), ‖.‖] is totally symmetric with (X, o) and (Y, o) are commutative groups then the group described in the Theorem2.10 is a topological group with respect to the topology induced from the norm.*

Proof. The proof is similar to the proof of the theorem2.20, so we omit the proof. ■

Theorem2.22. *Let the normed partial groupoid $[(G, X, Y, o), \|.\|]$ is symmetric about both of X and Y. The mapping $o{:}X\times Y\rightarrow G$ is a continuous bijection. Here the topology on $X\times Y$ is taken as the product topology induced from the topologies in X and Y induced from the norms restricted on X and Y.*

Proof. We see that:

(i) Since for each $a\in G$, the decomposition $a=a_1oa_2$, for $a_1 \in X$ and $a_2 \in Y$, so the mapping $o{:}X\times Y\rightarrow G$ defined as $(a_1,a_2)\mapsto a_1oa_2$ is bijective.

(ii) Let $a_1 \in X$, $a_2 \in Y$ and $r>0$, we consider the open ball $B_r(a_1oa_2)$ about $a_1oa_2 \in G$ in G.

Again we consider the open balls ${B_{r/2}}^X(a_1)$ and ${B_{r/2}}^Y(a_2)$ about $a_1 \in X$ and $a_2 \in Y$ in X and Y respectively.

So ${B_{r/2}}^X(a_1)\times{B_{r/2}}^Y(a_2)$ is a open set in $X\times Y$.

Let $(x_1,x_2)\in{B_{r/2}}^X(a_1)\times{B_{r/2}}^Y(a_2)$, i.e. $x_1 \in{B_{r/2}}^X(a_1)$ and $x_2 \in{B_{r/2}}^Y(a_2)$,

i.e. $\|x_1o{a_1}^{-1}\| <r/2$ and $\|x_2o{a_2}^{-1}\| <r/2$.

Now $x_1ox_2 \in G$ and $d(x_1ox_2,a_1oa_2)=\|(x_1o{a_1}^{-1})o(x_2o{a_2}^{-1})\| \leq \|x_1o{a_1}^{-1}\|+\|x_2o{a_2}^{-1}\| < r/2+r/2=r$,

so $x_1ox_2 \in B_r(a_1oa_2)$.

So $(x_1,x_2)\in{B_{r/2}}^X(a_1)\times{B_{r/2}}^Y(a_2) \Rightarrow x_1ox_2 \in B_r(a_1oa_2)$,

i.e. $o{:}X\times Y\rightarrow G$ is continuous.■

Theorem2.23. *A sequence $\{a^n\}_{n=1}^{\infty}$ in a normed partial groupoid $[(G, X, Y, o), \|.\|]$, where $a^n={a_1}^noa_2{}^n$ is convergent and converges to $a=a_1oa_2$ (as topological space) iff for any given $\varepsilon >0$, $\exists n_0 \in$ **N** such that $d(a^n,a) =\|({a_1}^no{a_1}^{-1})o({a_2}^no{a_2}^{-1})\| < \varepsilon$, $\forall\ n\geq n_0$, i.e. $d(a^n,a)\rightarrow 0$ as $n\rightarrow\infty$.*

Proof.The proof is easy as in metric topology, so we are omitting the proof.■

Theorem2.24. *Let $[(G, X, Y, o), \|.\|]$ is a normed partial groupoid. If $\{{a_1}^n\}_{n=1}^{\infty}$ and $\{{a_2}^n\}_{n=1}^{\infty}$ are convergent sequences in X and Y respectively converging to $a_1 \in X$ and $a_2 \in Y$ respectively, then the sequence $\{a^n\}_{n=1}^{\infty}$ in G where $a^n={a_1}^noa_2{}^n$ is convergent and converges to $a=a_1oa_2$.*

Proof. Let $\varepsilon >o$ be given. We consider the open balls ${B_{\varepsilon/2}}^X(a_1)$ and ${B_{\varepsilon/2}}^Y(a_2)$ about a_1 and a_2 in X and Y respectively. So $\exists$ n_1, $n_1 \in$ **N** such that ${a_1}^n \in{B_{\varepsilon/2}}^X(a_1)$, $\forall$ $n\geq n_1$ and ${a_2}^n \in{B_{\varepsilon/2}}^Y(a_2)$, $\forall$ $n\geq n_2$, i.e. $\|{a_1}^no{a_1}^{-1}\| < \varepsilon/2$ and $\|{a_2}^no{a_2}^{-1}\| < \varepsilon/2$.

So, $d(a^n,a)=\|({a_1}^no{a_1}^{-1})o({a_2}^no{a_2}^{-1})\| \leq \|{a_1}^no{a_1}^{-1}\|+ \|{a_2}^no{a_2}^{-1}\| < \varepsilon/2+\varepsilon/2=\varepsilon$, $\forall$ $n\geq=n_0=\max\{ n_1, n_2 \}$.

Hence the theorem follows.■

3 Operators on Partial Groupoids

Definition3.1. An operator T from the partial groupoid (G, X, Y, o) to (H, P, Q, *) is a mapping: $T{:}G\rightarrow H$

Satisfying the condition: $\forall$ $a_1 \in X$ and $a_2 \in Y$, $T(a_1)\in X$ and $T(a_2)\in Y$.

Now we consider the following definitions:

(a) The operator is said to be an homomorphism if $\forall$ $a_1 \in X$ and $a_2 \in Y$, $T(a_1 o a_2)=T(a_1)*T(a_2)$.

We define the set Hom(G,H)={ T : T:G$\rightarrow$H is a homomorphism }.

The operator is said to be an isomorphism if T is a bijective homorphism.

We define the set Iso(G,H)={ T : T:G$\rightarrow$H is an isomorphism }.

(b) If both the normed partial groupoids are K-normed partial groupoids, then the operator T is said to be K-linear if:
$\forall$ a$\in$G and $\lambda \in$K, $T(\lambda a)=\lambda T(a)$.

We define the sets L(G,H)={ T : T:G$\rightarrow$H is a K-linear operator } and LH(G,H)={ T : T:G$\rightarrow$H is a K-linear homomorphism }. So, LH(G,H)$\subseteq$L(G,H) and LH(G,H)$\subseteq$H(G,H).■

Remark3.2. For K-partial groupoids (G, X, Y, o) and (H, P, Q, *) with X$\cap$Y={e} and P$\cap$Q={$e^{/}$} we define T:G$\rightarrow$H, by T(a)=$e^{/}$ $\forall$ a$\in$G, then O$\in$Hom(G,H) and O$\in$L(G,H).■

Theorem3.3. *For partial groupoids (G, X, Y, o) and (H, P, Q, *) if T$\in$Hom(G,H), then T(e)=$e^{/}$.*

Proof. T(e)*$e^{/}$=T(e)=T(eoe)=T(e)*T(e). Now T(e)$\in$P and T(e)$\in$Q, so by left cancellation law in the group P or Q the theorem follows.■

Definition3.5. We consider the K-partial groupoids (G, X, Y, o) and (H, P, Q, *). We form the subsets A and B of LH(G,H) as for T$\in$LH(G,H), T$\in$A if T(G)$\subseteq$P and T$\in$B if T(G)$\subseteq$Q. Obviously O$\in$A and B both, also A$\cap$B={O}.

We define the operation $\times$ as follows:
(i) For T_1, $S_1 \in$A we define $T_1 \times S_1$:G$\rightarrow$H defined as $(T_1 \times S_1)(a)=T_1(a)*S_1(a)$.

(ii) For T_2, $S_2 \in$B we define $T_1 \times S_2$:G$\rightarrow$H defined as $(T_2 \times S_2)(a)=T_2(a)*S_2(a)$.

(iii) For $T_1 \in$A and $T_2 \in$B we define $T_1 \times T_2$:G$\rightarrow$H defined as $(T_1 \times T_2)(a)=T_1(a)*T_2(a)$.■

From the above definition we have the following theorem:

Theorem3.6. *(i) $T_1(a_2)=e^{/}$, $\forall$ $a_2 \in Y$. $T_1(a)=T_1(a_1)$, $\forall$ $a=a_1 o a_2 \in$. $T_1 \times S_1 \in A$, (ii) $T_2(a_1)=e^{/}$, $\forall$ $a_2 \in X$. $T_2(a)={}_1(a_2)$, $\forall$ $a=a_1 o a_2 \in$. $T_2 \times S_2 \in B$ (iii) $T_1 \times T_2 \in$LH(G,H). (iv) For each T$\in$LH(G,H) there exist unique $T_1 \in$A and $T_2 \in$B such that T= $T_1 \times T_2$.*

Proof.(i) (a) The first part is easy to prove since $T_1(a_2) \in$P$\cap$Q={$e^{/}$}.
(b) $T_1(a)=T_1(a_1 o a_1)$ =$T_1(a_1*T_1(a_2$ =$T_1(a_1*e^{/}$=$T_1(a_1$
(c) From the definition $T_1 \times S_1$ and the above fact it follows that $T_1 \times S_1$ is an operator. Now, $\forall$ $a_1 \in X$ and $a_2 \in Y$,

$(T_1\times S_1)(a_1oa_2 = T_1(a_1oa_2 * S_1(a_1oa_2$
$= T_1(a_1) * S_1(a_1) = (T_1(a_1) * S_1(a_1)) * (e^/ * e^/)$
$= (T_1(a_1) * S_1(a_1)) * (T_1(a_2) * S_1(a_2)) = (T_1\times S_1)(a_1) * (T_1\times S_1)(a_2)$.
Again, $\forall$ a$\in$G and $\lambda \in$K,
$(T_1\times S_1)(\lambda a) = T_1(\lambda a) * S_1(\lambda a) = (\lambda T_1(a)) * (\lambda S_1(a)) = \lambda(T_1\times S_1)$.
It is also easy to see that $(T_1\times S_1)(G) \subseteq P$.
So, $T_1\times S_1 \in A$.

(ii)The proof is same as that of (i).

(iii) $\forall$ $a_1 \in X$ and $a_2 \in Y$,
$(T_1\times T_2)(a_1) = (T_1\times T_2)(a_1oe) = T_1(a_1) * T_2(e) = T_1(a_1) * e^/ = T_1(a_1) \in P$. Similarly, $(T_1\times T_2)(a_2) \in Q$.
So, $T_1\times T_2$ is an operator.
$(T_1\times T_2)(a_1oa_2) = T_1(a_1) * T_2(a_2) = (T_1(a_1) * e^/) * (e^/ * T_2(a_2))$
$= (T_1(a_1) * T_2(a_1)) * (T_1(a_2) * T_2(a_2)) = (T_1\times T_2)(a_1) * (T_1\times T_2)(a_2)$.

Again, $\forall$ a$\in$G and $\lambda \in$K,
$(T_1\times S_1)(\lambda a) = T_1\times S_1(\lambda(a_1oa_2)) = T_1\times S_1(\lambda a_1 o \lambda a_2)$
$= (T_1)(\lambda a_1) * (T_2)(\lambda a_2) = \lambda T_1(a_1) * \lambda T_2(a_2)$
$= \lambda(T_1(a_1) * T_2(a_2)) = \lambda(T_1\times T_2(a)$.

So, $T_1\times S_1 \in LH(G,H)$.

(iv) For T$\in$LH(G,H), we define $T_1:G\rightarrow H$ and $T_2:G\rightarrow H$ defined by, $T_1(a) = T(a_1) \in P$ and $T_2(a) = T(a_2) \in Q$.
Now we see that, $T_1(a_1) = T(a_1) \in P$ and
$T_1(a_2) = T_1(eoa_2) = T(e) = e^/ \in Q$,
$T_1(a_1oa_2) = T(a_1) = T(a_1) * e^/ = T_1(a_1) * T_1(a_2)$.
Also, $T_1(G) \subseteq P$, by definition of T_1.
Hence, $T_1 \in A$.

Similarly, $T_2 \in B$.

Now, $T(a) = T(a_1oa_2) = T(a_1) * T(a_2) = T_1(a) * T_2(a)$
$= T_1(a_1) * T_2(a_2) = (T_1\times T_2)(a)$ i.e. $T = T_1\times T_2$.

To prove the uniqueness of T_1 and T_2, let $T = S_1\times S_2$, i.e. $T_1\times T_2 = S_1\times S_2$.
Now, for $a_1 \in X$, $(T_1\times T_2)(a_1) = (S_1\times S_2)(a_1)$
$\Rightarrow T_1(a_1) * T_2(a_1) = S_1(a_1) * S_2((a_1)$
$\Rightarrow T_1(a_1) = S_1(a_1) \Rightarrow T_1 = S_1$ on X, also $T_1 = S_1$ on Y (as they take the same value $e^/$ on Y)
Now, for $a = a_1oa_2 \in G$,
$T_1(a) = T_1(a_1oa_2) = T(a_1) = (S_1\times S_2)(a_1) = (S_1\times S_2)(a_1oe) = S_1(a_1) * S_2(e)$
$= S_1(a_1) * e^/ = S_1(a_1) * S_1(a_2) = S_1(a_1oa_1) = S_1(a)$,
i.e. $T_1 = S_1$.

Similarly, $T_2 = S_2$.

This shows the uniqueness of the decomposition.■

From the above discussion it is easy to deduce the following theorem:

Theorem3.7.*(LH(G,H), A, B, $\times$) is a partial groupoid.*
Proof. It is sufficient to prove that (A, $\times$) and (B, $\times$) are groups.

We see that: $\times$ is a binary operation on A and
(i) It is easy to check that for T_1, S_1, $R_1 \in$A, $(T_1\times S_1)\times R_1=T_1\times(S_1\times R_1)$.

(ii) O$\in$A is the identity element with respect to the binary operation $\times$ in A.

(iii) For $T_1 \in$A we define $(T_1)^{-1}$:G$\rightarrow$H, defined as $(T_1)^{-1}(a)=(T_1(a))^{-1}$.
Now, $\forall$ a_1 and $a_1 \in$Y,
$(T_1)^{-1}(a_1oa_2)=(T_1(a_1oa_2))^{-1}=(T_1(a_1)*T_1(a_2))^{-1}=(T_1(a_2))^{-1}*(T_1(a_1))^{-1}$
$=e^{/-1}*(T_1(a_1))^{-1}=(T_1(a_1))^{-1}*e^{/-1}=(T_1(a_1))^{-1}*(T_1(a_2))^{-1}*=(T_1)^{-1}(a_1)*(T_1)^{-1}(a_2)$

Again, $\forall$a$\in$G and $\lambda \in$K,
$(T_1)^{-1}(\lambda a)=(T_1(\lambda a))^{-1}=(T_1(\lambda a_1 o\lambda a_2)$
$=((T_1(\lambda a_1)o(T_1(\lambda a_1))^{-1}=((T_1(\lambda a_1)oe^{/})^{-1}=(T_1(\lambda a_1)^{-1}$
$=\lambda(T_1(a_1)^{-1}=\lambda(T_1(a)^{-1}=(\lambda T_1(a))^{-1}=\lambda(T_1)^{-1}(a)=\lambda(T_1)^{-1}(a)$

Also It is easy to see that $(T_1)^{-1}(G)\subseteq P$.
So, $(T_1)^{-1} \in$A.
It is easy to see that, $T_1\times(T_1)^{-1}=O=(T_1)^{-1}\times T_1$.
So, $(T_1)^{-1} \in$A is the invese element of $T_1 \in$A.
Hence (A, $\times$) is a group.
Similarly (B, $\times$) is also a group.
And the theorem follows.■

As similar above we can define the following definition:

Definition3.8. We consider the partial groupoids (G, X, Y, o) and (H, P, Q, *). We form the subsets M and N of H(G,H) as for T$\in$H(G,H), T$\in$M if T(G)$\subseteq$P and T$\in$N if T(G)$\subseteq$Q. Obviously O$\in$M and N both, also M$\cap$N={O}.

We define the operation $\times$ as follows:
(i) For T_1, $S_1\in$M we define $T_1\times S_1$:G$\rightarrow$H defined as $(T_1\times S_1)(a)=T_1(a)*S_1(a)$.

(ii) For T_2, $S_2\in$N we define $T_1\times S_2$:G$\rightarrow$H defined as $(T_2\times S_2)(a)=T_2(a)*S_2(a)$.

(iii) For $T_1\in$M and $T_2\in$N we define $T_1\times T_2$:G$\rightarrow$H defined as $(T_1\times T_2)(a)=T_1(a_1)*T_2(a_2)$.■

And we will have the following theorems:

Theorem3.9. *(i) $T_1\times S_1 \in M$, (ii) $T_2\times S_2 \in N$ (iii) $T_1\times T_2 \in H(G,H)$. (iv) For each T LH(G,H) there exist unique $T_1 M$ and $T_2 N$ such that $T= T_1\times T_2$.*■

Theorem3.10.*(H(G,H), M, N, $\times$) is partial groupoid.*■

It is easy to see that LH(G,H) is a non empty subset of H(G,H), (M, $\times$) and (N, $\times$) are subgroups of (A, $\times$) and (B, $\times$) and we have the following theorem for partial K-groupoids (G,X,Y,o) and (H,P,Q,*):

Theorem3.11.*(LH(G,H), A, B, $\times$) is a sub partial groupoid of the partial groupoid (H(G,H), M, N, $\times$).*■

Theorem3.12.*If T$\in$Iso(G,H), then T^{-1} $\in$Iso(H,G).*
Proof. The proof is easy, so we omitted the proof. ■

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